

# COMMENTS RELATED TO INFINITE WEDGE REPRESENTATIONS

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**ABSTRACT.** We study the infinite wedge representation and show how it is related to the universal extension of  $g[t, t^{-1}]$  the loop algebra of a complex semi-simple Lie algebra  $g$ . We also give an elementary proof of the boson-fermion correspondence. Our approach to proving this result is based on a combinatorial construction with partitions combined with an application of the Murnaghan-Nakayama rule.

## 1. INTRODUCTION

In this article, we make two remarks about the *infinite wedge representation*. To describe what we do let  $\mathfrak{gl}(\infty)$  denote the Lie algebra of  $\mathbb{Z} \times \mathbb{Z}$  *band infinite matrices*. Then  $\mathfrak{gl}(\infty)$  consists of those matrices  $A = (a_{ij})_{i,j \in \mathbb{Z}}$  with  $a_{ij} \in \mathbb{C}$  and  $a_{ij} = 0$  for all  $|i - j| \gg 0$ . Next let  $\widehat{\mathfrak{gl}}(\infty)$  denote the Lie algebra determined by the 2-cocycle  $c(\cdot, \cdot)$  of  $\mathfrak{gl}(\infty)$  with values in the trivial  $\mathfrak{gl}(\infty)$ -module  $\mathbb{C}$ :

$$c(A, B) := \sum_{i \leq 0, k > 0} a_{ik} b_{ki} - \sum_{i > 0, k \leq 0} a_{ik} b_{ki},$$

see for instance [1, p. 12] or [5, p. 115]. The *infinite wedge representation* is a suitably defined, see §4 for precise details, Lie algebra representation  $\rho : \widehat{\mathfrak{gl}}(\infty) \rightarrow \text{End}_{\mathbb{C}}(F)$ ; here  $F$  is the *infinite wedge space* that is the  $\mathbb{C}$ -vector space determined by the set  $\mathcal{S}$  which consists of those ordered strictly decreasing sequences of integers  $S = (s_1, s_2, \dots)$ ,  $s_i \in \mathbb{Z}$ , with the properties that  $s_i = s_{i-1} - 1$  for all  $i \gg 0$ .

To describe our first theorem, let  $g$  be a finite dimensional semi-simple complex Lie algebra,  $g[t, t^{-1}]$  its loop algebra and  $\widehat{g}$  the universal extension of  $g[t, t^{-1}]$  in the sense of Garland [3, §2] (see also [13, §7.9]). In Theorem 3.1 we show how the representation  $\rho$  is related to  $\widehat{g}$ .

Our second theorem, Theorem 6.2, gives an elementary proof of the *boson-fermion correspondence*, in the sense of Kac-Raina-Rozhkovskaya [6, Lecture 5, p. 46]. To place this theorem in its proper context, let  $\mathfrak{s}$  denote the *oscillator algebra*, which is the universal extension of  $\mathbb{C}[t, t^{-1}]$  the loop algebra of the abelian Lie algebra  $\mathbb{C}$ . The Lie algebra  $\mathfrak{s}$  is faithfully represented in  $\text{End}_{\mathbb{C}}(F)$  and also in  $\text{End}_{\mathbb{C}}(B)$ , where  $B$  denotes the polynomial ring in countably many variables with coefficients in the ring of Laurent polynomials. The boson-fermion correspondence, as formulated in [6, Lecture 5, p. 46] compare also with [5,

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§14.9–14.10], concerns extending these representations to all of  $\widehat{\mathfrak{gl}}(\infty)$  in such a way that an evident  $\mathbb{C}$ -linear isomorphism  $F \rightarrow B$  becomes an isomorphism of  $\widehat{\mathfrak{gl}}(\infty)$ -modules; see §6 for more precise details. The traditional approach for proving this result is by way of vertex-operators, see [5] and [6, Lecture 6, p. 46]. The key point to our approach, which does not require the use of vertex-operators, is a combinatorial construction related to partitions, see §5, together with the Murnaghan-Nakayama rule which we recall in §2.6.

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## 2. PRELIMINARIES

In this section, to fix notation and terminology for what follows, we recall a handful of combinatorial and Lie theoretic concepts. For the most part we use combinatorial terminology and conventions similar to that of [7, I §1 – 5] and Lie theoretic terminology and conventions similar to that of [5, §7 and §14].

**2.1.** Let  $\mathcal{P}$  denote the set of partitions. Then  $\mathcal{P}$  consists of those infinite weakly decreasing sequences of non-negative integers  $\lambda = (\lambda_1, \lambda_2, \dots)$  with the property that at most finitely many of the  $\lambda_i$  are nonzero. If  $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}$ , then the number  $\text{weight}(\lambda) := \sum_{i=1}^{\infty} \lambda_i$  is called the *weight* of  $\lambda$  and we denote by  $\mathcal{P}_d$  the set of partitions of weight  $d$ . If  $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}$ , then we often identify  $\lambda$  with the finite weakly decreasing sequence  $(\lambda_1, \dots, \lambda_r)$ , where  $r = \text{length}(\lambda) := \max\{i : \lambda_i \neq 0\}$ .

**2.2.** The *Young diagram* of a partition  $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}$  is defined to be the set of points  $(i, j) \in \mathbb{Z}^2$  such that  $1 \leq j \leq \lambda_i$ . When drawing the Young diagram associated to a partition we use the convention that the first coordinate is the row index, starts at 1 and increases from left to right. Similarly, the second coordinate is the column index, starts at 1 and increases downward. We refer to the elements  $(i, j)$  of a Young diagram as the *boxes* of the associated partition and the entire  $i$  and  $j$  as the sides of the box.

**2.3.** If  $\lambda, \mu \in \mathcal{P}$  and  $\lambda \supseteq \mu$ , so  $\lambda_i \geq \mu_i$  for all  $i \geq 1$ , then the set theoretic difference of the Young diagrams corresponding to  $\lambda$  and  $\mu$  is denoted by  $\lambda \setminus \mu$  and is called a *skew diagram*. If  $\lambda, \mu \in \mathcal{P}$  and  $\lambda \supseteq \mu$ , then let  $\theta := \lambda \setminus \mu$  denote the skew diagram that they determine. By a *path* in  $\theta$ , we mean a sequence  $x_0, x_1, \dots, x_m$  with  $x_i \in \theta$ , such that  $x_{i-1}$  and  $x_i$  have a common side for  $1 \leq i \leq m$ . A subset  $\nu \subseteq \theta$  is said to be *connected* if every two boxes in  $\nu$  can be connected by a path in  $\nu$ . The *length* of  $\theta$  is defined to be the number of boxes that appear in its diagram and is denoted by  $\#\theta$ . We say that  $\theta$  is a *border strip* if it is connected and if it contains no  $2 \times 2$  box. Finally, if  $\theta$  is a border strip, then we denote its *height* by  $\text{height}(\theta)$  and define it to be one less than the number of rows that it occupies.

**2.4.** The symmetric group  $S_n$  acts on the polynomial ring  $\mathbb{C}[x_1, \dots, x_n]$  by permuting the variables and we let  $\Lambda_n$  denote the subring of invariants. We then have that  $\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$  where  $\Lambda_n^k \subseteq \Lambda_n$  is the subspace of symmetric polynomials of degree  $k$ . If  $k \in \mathbb{Z}_{\geq 0}$ ,  $m, n \in \mathbb{Z}_{\geq 1}$  and  $m \geq n$ , we have evident restriction maps  $\rho_{m,n}^k : \Lambda_m^k \rightarrow \Lambda_n^k$ ; let  $\Lambda^k = \varprojlim \Lambda_n^k$  and  $\Lambda = \bigoplus_{k \geq 0} \Lambda^k$ . Then  $\Lambda = \mathbb{C}[h_1, h_2, \dots]$  where the  $h_k$  are such that their image in  $\Lambda_n^k$  is the  $k$ th complete symmetric function in the variables  $x_1, \dots, x_n$ .

**2.5.** Let  $H(Z) := \sum_{k \geq 0} h_k Z^k \in \Lambda[[Z]]$  and define  $p_k \in \Lambda$  by the coefficient of  $Z^{k-1}$  in the power series  $P(Z) := H'(Z)/H(Z)$ . The image of each  $p_k$  in  $\Lambda_n^k$  is the  $k$ th power sum in the variables  $x_1, \dots, x_n$  and the  $h_k$  can be expressed in terms of the  $p_k$  via the equality of power series  $H(Z) = \exp(\sum_{k \geq 1} t_k Z^k)$ ; here  $t_k = p_k/k$ .

The Schur functions  $s_\lambda$ , defined for all partitions  $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}$ , are defined by  $s_\lambda := \det(h_{\lambda_i - i + j})_{1 \leq i, j \leq n}$ , where  $n = \text{length}(\lambda)$ , and form a  $\mathbb{C}$ -basis for  $\Lambda$ . In what follows we let  $\langle \cdot, \cdot \rangle$  denote the symmetric bilinear form on  $\Lambda$  for which the Schur polynomials are orthonormal. In particular,  $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$ .

**2.6.** By abuse of notation, we let  $p_k \in \text{End}_{\mathbb{C}}(\Lambda)$  be the  $\mathbb{C}$ -linear endomorphism given by multiplication by  $p_k$ . The adjoint of  $p_k$  with respect to  $\langle \cdot, \cdot \rangle$ , which we denote by  $p_k^\perp$ , is the  $\mathbb{C}$ -linear endomorphism given by the differential operator  $k \frac{\partial}{\partial p_k}$ , [7, p. 76].

The effect of the operator  $p_k$  in the basis of Schur polynomials is given by the Murnaghan-Nakayama rule:

$$(2.1) \quad p_k s_\lambda = \sum_{\substack{\nu \supseteq \lambda, \\ \nu \setminus \lambda \text{ is a border strip} \\ \text{of length } k}} (-1)^{\text{height}(\nu \setminus \lambda)} s_\nu,$$

[10, p. 601]. Using (2.1), in conjunction with [7, I.V. Ex. 3, p. 75], we deduce the adjoint form of the Murnaghan-Nakayama rule:

$$(2.2) \quad p_k^\perp s_\lambda = \sum_{\substack{\lambda \supseteq \nu, \\ \lambda \setminus \nu \text{ is a border strip} \\ \text{of length } k}} (-1)^{\text{height}(\lambda \setminus \nu)} s_\nu.$$

**2.7.** We let  $\text{Mat}(\infty)$  denote the  $\mathbb{C}$ -vector space of  $\mathbb{Z} \times \mathbb{Z}$  matrices with entries in  $\mathbb{C}$ . If  $A = (a_{ij})_{i,j \in \mathbb{Z}}, B = (b_{ij})_{i,j \in \mathbb{Z}} \in \text{Mat}(\infty)$  and  $a_{ik} b_{kj} = 0$ , for all  $i, j \in \mathbb{Z}$  and almost all  $k \in \mathbb{Z}$ , then their product is given by  $C = AB := (c_{ij})_{i,j \in \mathbb{Z}}$ , where  $c_{ij} = \sum_{k \in \mathbb{Z}} a_{ik} b_{kj}$ .

**2.8.** We let  $E_{ij}$  denote the element of  $\text{Mat}(\infty)$  with  $i, j$  entry equal to 1 and all other entries equal to zero. We say that a matrix  $A = (a_{ij})_{i,j \in \mathbb{Z}} \in \text{Mat}(\infty)$  is a *band infinite matrix* if  $a_{ij} = 0$  for all  $|i - j| \gg 0$ . We denote the collection of band infinite matrices by  $\mathbf{gl}(\infty)$  and regard it as a Lie algebra with Lie bracket given by  $[A, B] = AB - BA$ . We often

express elements of  $\mathfrak{gl}(\infty)$  as infinite sums of matrices. For example, the identity matrix  $1_{\mathbb{Z} \times \mathbb{Z}} = (\delta_{ij})_{i,j \in \mathbb{Z}}$ , can be expressed as  $1_{\mathbb{Z} \times \mathbb{Z}} = \sum_{p \in \mathbb{Z}} E_{pp}$ . Also every element of  $\mathfrak{gl}(\infty)$  can be written as a finite linear combination of matrices of the form  $\sum_i a_i E_{i,i+k}$ , where  $k \in \mathbb{Z}$  and  $a_i \in \mathbb{C}$ .

**2.9.** Let  $\mathfrak{gl}_N[t, t^{-1}] := \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{gl}_N(\mathbb{C})$  which we regard as a Lie algebra with Lie bracket determined by

$$[f(t) \otimes A, g(t) \otimes B] = f(t)g(t) \otimes [A, B].$$

If  $t^m \otimes e_{ij}$ , for  $i, j = 1, \dots, N$  and  $m \in \mathbb{Z}$ , denotes the standard basis elements of  $\mathfrak{gl}_N[t, t^{-1}]$ , we then have that

$$[t^m \otimes e_{ij}, t^n \otimes e_{kl}] = t^{m+n} \otimes (\delta_{jk} e_{il} - \delta_{li} e_{kj}),$$

and that the map

$$(2.3) \quad \iota_N : \mathfrak{gl}_N[t, t^{-1}] \rightarrow \mathfrak{gl}(\infty),$$

determined by

$$t^m \otimes e_{ij} \mapsto \sum_{k \in \mathbb{Z}} E_{N(k-m)+i, Nk+j},$$

is a monomorphism of Lie algebras. The image of  $\iota_N$  is the Lie algebra of *N-periodic band infinite matrices*, that is those  $\mathbb{Z} \times \mathbb{Z}$  band infinite matrices  $A = (a_{ij})_{i,j \in \mathbb{Z}}$  for which  $a_{i+N,j+N} = a_{ij}$ , for all  $i, j \in \mathbb{Z}$ .

**2.10.** To define the Lie algebra  $\widehat{\mathfrak{gl}}(\infty)$ , first let

$$J := \sum_{m \leq 0} E_{mm} - \sum_{m > 0} E_{mm} \in \mathfrak{gl}(\infty)$$

and observe that if  $A$  and  $B$  are elements of  $\mathfrak{gl}(\infty)$ , then the matrix  $[J, A]B$  has at most finitely many nonzero diagonal elements. Also, and the expression  $\frac{1}{2} \text{tr}([J, A]B)$  is a well defined element of  $\mathbb{C}$ . In particular, we have

$$(2.4) \quad \frac{1}{2} \text{tr}([J, A]B) = \sum_{i \leq 0, k > 0} a_{ik} b_{ki} - \sum_{i > 0, k \leq 0} a_{ik} b_{ki},$$

and we define the Lie algebra  $\widehat{\mathfrak{gl}}(\infty)$  to be the central extension determined by the following 2-cocycle of  $\mathfrak{gl}(\infty)$  with values in the trivial  $\mathfrak{gl}(\infty)$ -module  $\mathbb{C}$ :

$$(2.5) \quad c(A, B) := \frac{1}{2} \text{tr}([J, A]B) = \sum_{i \leq 0, k > 0} a_{ik} b_{ki} - \sum_{i > 0, k \leq 0} a_{ik} b_{ki}.$$

As a special case of (2.5), we have that

$$(2.6) \quad c(E_{ij}, E_{k\ell}) = \begin{cases} -1 & i = \ell > 0, j = k \leq 0 \\ 1 & i = \ell \leq 0, j = k > 0 \\ 0 & \text{otherwise,} \end{cases}$$

for  $i, j, k, \ell \in \mathbb{Z}$ ; compare with [1, p. 12] or [5, p. 115 and p. 313]. Explicitly, as a  $\mathbb{C}$ -vector space

$$\widehat{\mathfrak{gl}}(\infty) = \mathbb{C} \oplus \mathfrak{gl}(\infty),$$

and the Lie bracket is defined by

$$[(a, x), (b, y)] = (c(x, y), [x, y]),$$

for all  $(a, x), (b, y) \in \widehat{\mathfrak{gl}}(\infty)$ .

**2.11.** We regard  $\mathbb{C}[t, t^{-1}]$ , the ring of Laurent polynomials, as the loop algebra of the abelian Lie algebra  $\mathbb{C}$ . The *oscillator algebra* is the Lie algebra  $\mathfrak{s}$  determined by the 2-cocycle with values in the trivial  $R$ -module  $\mathbb{C}$  given by:

$$\omega : \mathbb{C}[t, t^{-1}] \times \mathbb{C}[t, t^{-1}] \rightarrow \mathbb{C},$$

$$\omega((f(t), g(t))) := \text{res} \left( \frac{df}{dt} g \right).$$

Concretely,

$$\mathfrak{s} = \mathbb{C} \oplus \mathbb{C}[t, t^{-1}],$$

and the bracket is given by

$$[(a, t^m), (b, t^n)] = (m\delta_{m, -n}, 0),$$

for all  $a, b \in \mathbb{C}$  and  $m, n \in \mathbb{Z}$ .

**2.12.** As in [5, p. 313], we realize the oscillator algebra  $\mathfrak{s}$  as a subalgebra of  $\widehat{\mathfrak{gl}}(\infty)$  by the monomorphism of Lie algebras

$$(2.7) \quad \delta_0 : \mathfrak{s} \rightarrow \widehat{\mathfrak{gl}}(\infty),$$

defined by

$$(a, t^m) \mapsto \left( a, \sum_{j \in \mathbb{Z}} E_{j, j+m} \right).$$

### 3. THE LIE ALGEBRA $\widehat{\mathfrak{gl}}(\infty)$ AND UNIVERSAL EXTENSIONS

In this section we establish Theorem 3.1 which shows how the Lie algebra  $\widehat{\mathfrak{gl}}(\infty)$  is related to the Lie algebra  $\widehat{g}$  which we define to be the *universal extension* of  $g[t, t^{-1}]$  the Loop algebra of  $g$  a complex finite dimensional semi-simple Lie algebra.

**3.1.** Let  $g$  be a complex finite dimensional semi-simple Lie algebra and  $\kappa(\cdot, \cdot)$  its killing form. We denote by  $\widehat{g}$  the *universal extension* of  $g[t, t^{-1}]$ . Then  $\widehat{g}$  is the central extension determined by the 2-cocycle

$$(3.1) \quad u(\cdot, \cdot) : g[t, t^{-1}] \times g[t, t^{-1}] \rightarrow \mathbb{C}$$

defined by

$$(3.2) \quad u \left( \sum t^i \otimes x_i, \sum t^j \otimes y_j \right) := \sum i \kappa(x_i, y_{-i}),$$

[3, §2] see also [13, §7.9] especially [13, §7.9.6, p. 250].

To relate  $\widehat{g}$  and  $\widehat{\mathfrak{gl}}(\infty)$ , we choose a basis for  $g$  and then consider its *extended adjoint representation*:

$$(3.3) \quad 1 \otimes \text{ad} : g[t, t^{-1}] \rightarrow \mathfrak{gl}(\infty),$$

see (3.6) below.

The morphism  $1 \otimes \text{ad}$ , given by (3.3), allows us to compare the pullback of  $\widehat{\mathfrak{gl}}(\infty)$ , with respect to  $1 \otimes \text{ad}$ , and the universal extension  $\widehat{g}$ . In §3.3, we prove:

**Theorem 3.1.** *The universal extension of  $g[t, t^{-1}]$  is the pull-back of  $\widehat{\mathfrak{gl}}(\infty)$  via  $1 \otimes \text{ad}$ , the extended adjoint representation of  $g$ .*

**3.2.** Before proving Theorem 3.1 we first observe:

**Proposition 3.2.** *The pullback of  $c(\cdot, \cdot)$  to  $\mathfrak{gl}_N[t, t^{-1}]$  via  $\iota_N$  is given by:*

$$(3.4) \quad c(\iota_N(t^m \otimes x), \iota_N(t^n \otimes y)) = m \delta_{m, -n} \text{tr}(xy).$$

*Proof.* In light of the map (2.3), it suffices to check that, for fixed  $N \in \mathbb{Z}_{\geq 1}$ ,  $1 \leq i, j, k, \ell \leq N$ ,  $m, n \in \mathbb{Z}$ , we have

$$c \left( \sum_{p \in \mathbb{Z}} E_{N(p-m)+i, Np+j}, \sum_{q \in \mathbb{Z}} E_{N(q-n)+k, Nq+\ell} \right) = \begin{cases} m & \text{if } j = k, i = \ell \text{ and } m = -n. \\ 0 & \text{otherwise.} \end{cases}$$

To compute

$$c \left( \sum_{p \in \mathbb{Z}} E_{N(p-m)+i, Np+j}, \sum_{q \in \mathbb{Z}} E_{N(q-n)+k, Nq+\ell} \right),$$

considering (2.5), it is clear that we need to understand the quantity:

$$(3.5) \quad \sum_{\substack{p, q \in \mathbb{Z}, \\ Np+j > 0, \\ k+N(q-n) > 0, \\ i+N(p-m) \leq 0, \\ Nq+\ell \leq 0}} \delta_{Np+j, N(q-n)+k} \delta_{N(p-m)+i, Nq+\ell} - \sum_{\substack{p, q \in \mathbb{Z}, \\ Np+j \leq 0, \\ k+N(q-n) \leq 0, \\ i+N(p-m) > 0, \\ Nq+\ell > 0}} \delta_{Np+j, N(q-n)+k} \delta_{N(p-m)+i, Nq+\ell}.$$

To this end, we make the following deductions:

(a) if (3.5) is nonzero, then  $m = -n$ ;

- (b) if  $m \geq 0$  the first sum appearing in (3.5) is nonzero if and only if  $j = k$  and  $i = \ell$ , while the second sum is zero; the nonzero summands appearing (3.5), when  $j = k$  and  $i = \ell$ , are in bijection with the set of pairs  $(p, q) \in \mathbb{Z} \times \mathbb{Z}$  with  $-1 \leq p \leq m$  and  $n \leq q < 0$ ;
- (c) if  $m < 0$ , the second sum appearing in (3.5) is nonzero if and only if  $j = k$  and  $i = \ell$ , while the first sum is zero; the nonzero summands appearing in (3.5), when  $j = k$  and  $i = \ell$ , are in the bijection with the set of pairs  $(p, q) \in \mathbb{Z} \times \mathbb{Z}$  with  $m \leq p < 0$ ,  $0 \leq q < n$ .

The conclusion of Proposition 3.2 follows immediately from these deductions.  $\square$

**3.3.** We now establish Theorem 3.1. To do so, first consider an arbitrary semi-simple Lie algebra  $g$  and its adjoint representation

$$\text{ad} : g \rightarrow \text{End}_{\mathbb{C}}(g).$$

Let  $N = \dim_{\mathbb{C}} g$  and fix a basis for  $g$ . By composition we obtain a representation

$$\text{ad} : g \rightarrow \text{End}_{\mathbb{C}}(g) \xrightarrow{\sim} \mathfrak{gl}_N(\mathbb{C}),$$

which we can use to define the *extended adjoint representation* of  $g$

$$(3.6) \quad g[t, t^{-1}] \xrightarrow{1 \otimes \text{ad}} \mathfrak{gl}_N(\mathbb{C})[t, t^{-1}] \xrightarrow{\iota_N} \mathfrak{gl}(\infty).$$

The homomorphism (3.6) allows us to compare the pull-back of  $\widehat{\mathfrak{gl}}(\infty)$ , via  $1 \otimes \text{ad}$ , with  $\widehat{g}$ .

*Proof of Theorem 3.1.* It is enough to show that

$$u \left( \sum t^i \otimes x_i, \sum t^j \otimes y_j \right) := \sum i\kappa(x_i, y_{-i})$$

equals

$$c \left( \sum t^i \otimes \text{ad } x_i, \sum t^j \otimes \text{ad } y_j \right).$$

That this equality holds true follows from the fact that  $\kappa(x, y) := \text{tr}(\text{ad } x \text{ad } y)$  and from Proposition 3.2.  $\square$

#### 4. SEMI-INFINITE MONOMIALS AND THE INFINITE WEDGE REPRESENTATION

In this section we study certain subsequences of  $\mathbb{Z}$  which we refer to as *semi-infinite monomials*, see §4.1. We then describe the *infinite wedge space* and the *infinite wedge representation* of the Lie algebra  $\widehat{\mathfrak{gl}}(\infty)$ , see §4.6 and §4.8 respectively. What we do here is influenced heavily by what is done in [5], [12], [4] and [8]. We give proofs of all assertions for completeness and because they are needed in our proof of Theorem 6.2.

**4.1.** By a *semi-infinite monomial* we mean an ordered strictly decreasing sequence of integers  $S = (s_1, s_2, \dots)$ ,  $s_i \in \mathbb{Z}$ , with the properties that  $s_i = s_{i-1} - 1$  for all  $i \gg 0$ . We let  $\mathcal{S}$  denote

the set of semi-infinite monomials. If  $S \in \mathcal{S}$ , then define strictly decreasing sequences of integers  $S_+$  and  $S_-$  by  $S_+ := S \setminus \mathbb{Z}_{\leq 0}$  and  $S_- := \mathbb{Z}_{\leq 0} \setminus S$ .

**4.2.** If  $S = (s_1, s_2, \dots) \in \mathcal{S}$ , then there exists a unique integer  $m$  with the property that  $s_i = m - i + 1$  for all  $i \gg 0$ . We refer to this number as the *charge* of  $S$  and denote it by  $\text{charge}(S)$ , compare with [12, p. 12], [5, p. 310], and [9, A.3] for instance. If  $m \in \mathbb{Z}$ , then let  $\mathcal{S}_m := \{S \in \mathcal{S} : \text{charge}(S) = m\}$ . We record the following proposition for later use.

**Proposition 4.1.** *The following assertions hold true:*

- (a) *If  $S \in \mathcal{S}$ , then  $\text{charge}(S) = \#S_+ - \#S_-$ ;*
- (b) *Let  $m \in \mathbb{Z}$ . The map  $\lambda : \mathcal{S}_m \rightarrow \mathcal{P}$  defined by*

$$S = (s_1, s_2, \dots) \mapsto \lambda(S) = (\lambda_1, \lambda_2, \dots),$$

*where*

$$(4.1) \quad \lambda_j := s_j - m + j - 1,$$

*is a bijection.*

*Proof.* To prove (a), let

$$(4.2) \quad S := (s_1, s_2, \dots) \in \mathcal{S}, S_+ := (s_1, \dots, s_\ell) \text{ and } S_- := (n_1, \dots, n_r);$$

here  $s_1 > s_2 > \dots > s_\ell$  and  $0 \geq n_1 > n_2 > \dots > n_r$ . Considering the definitions of  $S_+$  and  $S_-$  we deduce that

$$(4.3) \quad s_{n+k} = n_r - k \text{ for } n := \ell - n_r - r + 1 \text{ and } k \geq 1.$$

Now suppose that  $m := \ell - r = \#S_+ - \#S_-$  and let  $i = n + k$  for  $k \geq 1$ . We then have  $m - i + 1 = n_r - k$  which equals  $s_i$  by (4.3). Conversely, suppose that  $s_i = m - i + 1$  for all  $i \gg 0$ . We then have for all  $k \gg 0$  that

$$(4.4) \quad s_{n+k} = m - n - k + 1 = m - \ell + n_r + r - k.$$

Combining (4.3) and (4.4), we then have

$$(4.5) \quad n_r - k = m - \ell + n_r + r - k$$

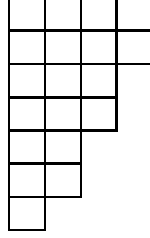
and so  $m = \ell - r$  as desired.

For (b), first note that the map  $\lambda$  is clearly injective. To see that it is surjective, if  $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}$ , then define an element  $S = (s_1, s_2, \dots) \in \mathcal{S}_m$  by  $s_j = \lambda_j + m - j + 1$ . By construction  $S \in \mathcal{S}$ . To see that  $S \in \mathcal{S}_m$  note that  $s_j = m - j + 1$  for  $j > \text{length}(\lambda)$ .  $\square$

**4.3. Remark.** When we express  $S \in \mathcal{S}$  as in (4.2), the length of the partition  $\lambda(S)$  equals the number  $n$  defined in (4.3). Also the weight of the partition  $\lambda(S)$  is sometimes referred to as the *energy* of  $S$ , [5, p. 310].



**4.4. Example.** We can use the approach of [11, §7.2] to give a graphical interpretation of Proposition 4.1 for the case  $m = 0$ . The case  $m \neq 0$  can be handled similarly with a shift. As an example, the Young diagram associated to the partition  $\lambda = (4, 4, 3, 3, 2, 2, 1) \in \mathcal{P}_{19}$  is:



If we cut this Young diagram along the main diagonal then there are 3 rows in the top piece and 3 columns in the bottom piece. Let  $u_i$ ,  $i = 1, 2, 3$ , denote the number of boxes in the  $i$ -th row of the top piece and let  $v_i$ ,  $i = 1, 2, 3$ , denote the number of boxes in the  $i$ -th column of the bottom piece. Then,  $u_1 = 3.5$ ,  $u_2 = 2.5$ ,  $u_3 = .5$  and  $v_1 = 6.5$ ,  $v_2 = 4.5$ , and  $v_3 = 1.5$ .

If  $S$  is the charge zero semi-infinite monomial corresponding to  $\lambda$ , then  $S$  is determined by the condition that

$$S_+ = (u_1 + .5, u_2 + .5, u_3 + .5) = (4, 3, 1)$$

and

$$S_- = (-v_3 + .5, -v_2 + .5, -v_1 + .5) = (-1, -4, -6).$$

In other words,

$$S = (4, 3, 1, 0, -2, -3, -5, -7, -8, \dots)$$

is the element of  $\mathcal{S}_0$  corresponding to the partition  $\lambda = (4, 4, 3, 3, 2, 2, 1)$ .

We can also relate the set  $S$  to the *code*, in the sense of [2, §2], of the partition  $\lambda$ . Specifically, if  $n \in \mathbb{Z}$ ,  $n \geq 1$  and  $n \notin S$ , then  $n$  corresponds to an  $R$ ; if  $n \geq 1$  and  $n \in S$ , then  $n$  corresponds to a  $U$ . If  $n \in \mathbb{Z}$ ,  $n \leq 0$  and  $n \in S$ , then  $n$  corresponds to a  $U$ ; if  $n \leq 0$  and  $n \notin S$ , then  $n$  corresponds to an  $R$ . The string consisting of these  $R$ 's and  $U$ 's is the code corresponding to  $\lambda$  and our set  $S$ .

**4.5.** Let  $\lambda : \mathcal{S} \rightarrow \mathcal{P}$  denote the extension of the bijections  $\lambda : \mathcal{S}_m \rightarrow \mathcal{P}$  described in Proposition 4.1 (b). Also, to keep track of various minus signs which appear in what follows, we make the following definition: if  $S \in \mathcal{S}$  and  $j \in \mathbb{Z}$ , then define  $\text{count}(j, S)$  to be the number of elements of  $S$  that are strictly greater than  $j$ , that is:

$$(4.6) \quad \text{count}(j, S) := \#\{s \in S : j < s\}.$$

**4.6.** The *infinite wedge space* is the  $\mathbb{C}$ -vector space  $F := \bigoplus_{S \in \mathcal{S}} \mathbb{C}$  determined by the set  $\mathcal{S}$ , see for instance [5, §14.15] or [9, p. 76]. In particular,

$$F = \text{span}_{\mathbb{C}}\{v_S : S \in \mathcal{S}\}$$

where  $v_S = (r_T)_{T \in \mathcal{S}}$  denotes the element of  $F$  given by  $r_T = 0$  for  $T \neq S$  and  $r_S = 1$ . If  $m \in \mathbb{Z}$ , then let  $F^{(m)} := \text{span}_{\mathbb{C}}\{v_S : S \in \mathcal{S}_m\}$ . We then have  $F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}$ , compare with [5, p. 310].

**4.7.** We now recall the definition of *wedging* and *contracting* operators. Our approach here is only notationally different from that of [5, p. 311]. On the other hand, we find our approach useful for relating these operators to our combinatorial construction on partitions, see §5 and especially Proposition 5.2.

To begin with, if  $S = (s_1, s_2, \dots)$  is an ordered strictly decreasing sequence of integers and  $j \in \mathbb{Z}$ , then we use the notations  $S \cup \{j\}$  and  $S \setminus \{j\}$  to denote the ordered strictly decreasing sequence of integers determined by the sets  $\{s_1, s_2, \dots\} \cup \{j\}$  and  $\{s_1, s_2, \dots\} \setminus \{j\}$  respectively.

Next, given  $j \in \mathbb{Z}$ , define elements  $f_j, f_j^* \in \text{End}_{\mathbb{C}}(F)$ , for  $j \in \mathbb{Z}$ , by:

$$(4.7) \quad f_j(v_S) := \begin{cases} (-1)^{\text{count}(j, S)} v_{S \cup \{j\}} & \text{if } j \notin S \\ 0 & \text{if } j \in S \end{cases}$$

and

$$(4.8) \quad f_j^*(v_S) := \begin{cases} (-1)^{\text{count}(j, S)} v_{S \setminus \{j\}} & \text{if } j \in S \\ 0 & \text{if } j \notin S, \end{cases}$$

and extending  $\mathbb{C}$ -linearly, compare with [5, §14.17], [1, p. 12], and [1, §A]. These endomorphisms have the properties that

$$(4.9) \quad f_i f_j^* + f_j^* f_i = \delta_{ij}, \quad f_i f_j + f_j f_i = 0, \quad f_i^* f_j^* + f_j^* f_i^* = 0$$

and

$$(4.10) \quad [f_i f_j^*, f_\ell f_k^*] = \delta_{j\ell} f_i f_k^* - \delta_{ik} f_\ell f_j^*,$$

for all  $i, j, k, \ell \in \mathbb{Z}$ , see [5, p. 311] for example.

For completeness, we note that (4.9) follows immediately from the definitions given in (4.7) and (4.8). On the other hand, (4.10) is a consequence of (4.9). Indeed, first note:

$$[f_i f_j^*, f_\ell f_k^*] = f_i f_j^* f_\ell f_k^* - f_i f_\ell f_k^* f_j^* + f_i f_\ell f_k^* f_j^* - f_\ell f_k^* f_i f_j^*$$

which can be rewritten using the second and third properties of (4.9) as:

$$(4.11) \quad f_i(f_j^* f_\ell + f_\ell f_j^*) f_k^* - f_\ell(f_i f_k^* + f_k^* f_i) f_j^*.$$

Applying the first property given in (4.9) to (4.11) yields the righthand side of (4.10).

Note also that the operators  $f_i$ , for  $i \in \mathbb{Z}$ , map  $F^{(m)}$  to  $F^{(m+1)}$ , the operators  $f_i^*$ , for  $i \in \mathbb{Z}$ , map  $F^{(m)}$  to  $F^{(m-1)}$  whereas the operators  $f_i f_j^*$ , for  $i, j \in \mathbb{Z}$ , map  $F^{(m)}$  to  $F^{(m)}$ .

**4.8.** The *infinite wedge representation* is the Lie algebra homomorphism

$$\rho : \widehat{\mathfrak{gl}}(\infty) \rightarrow \text{End}_{\mathbb{C}}(F)$$

determined by the conditions that

$$(4.12) \quad \rho((0, E_{ij})) = \begin{cases} f_i f_j^* & \text{if } i \neq j \text{ or } i = j > 0 \\ f_i f_i^* - \text{id}_F & \text{if } j = i \leq 0 \end{cases}$$

and

$$(4.13) \quad \rho((a, 0)) = a \text{id}_F,$$

for  $a \in \mathbb{C}$ , compare with [5, p. 313] for instance.

The fact that the above conditions (4.12) and (4.13) determine a representation of Lie algebras is deduced easily from property (4.10) above together with the definition of the 2-cocycle  $c(\cdot, \cdot)$ , given in (2.6), and the fact that every element of  $\widehat{\mathfrak{gl}}(\infty)$  can be written as a finite linear combination of matrices of the form  $\sum_{i \in \mathbb{Z}} a_i E_{i, i+k}$ , where  $k \in \mathbb{Z}$  and  $a_i \in \mathbb{C}$ .

**4.9.** In what follows we refer to the restriction of  $\rho$  to the image of the morphism (2.7) as the *infinite wedge representation of the oscillator algebra*  $\mathfrak{s}$ .

## 5. COMBINATORIAL PROPERTIES OF THE OPERATORS $f_i f_j^*$

In this section we define and study certain operators on partitions. This construction will be used in our definition of the bosonic representation of the Lie algebra  $\widehat{\mathfrak{gl}}(\infty)$ , see §6. Our main result is Proposition 5.2 which describes the combinatorics encoded in the vector

$$(5.1) \quad f_i f_j^*(v_S) = (-1)^\alpha v_T;$$

here

$$S := (s_1, s_2, \dots) \in \mathcal{S},$$

$i, j \in \mathbb{Z}$ , are such that

$$j \in S \text{ and } i \notin S \setminus \{j\},$$

$$T := (S \setminus \{j\}) \cup \{i\},$$

and

$$\alpha := \text{count}(i, S \setminus \{j\}) - \text{count}(j, S).$$

As it turns out the combinatorics encoded in (5.1) are related to a certain skew diagram associated to the partition determined by  $S$ , see Proposition 5.1 and Proposition 5.2.

**5.1.** Let  $m, i \in \mathbb{Z}$ , let  $\mathcal{P}_{m,i}$  denote the set

$$\mathcal{P}_{m,i} := \{\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P} : \lambda_k \neq i - m + k - 1 \text{ for all } k\},$$

and let  $\mathcal{P}_{m,i}^*$  denote the set

$$\mathcal{P}_{m,i}^* := \{\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P} : \lambda_k = i - m + k - 1 \text{ for some } k\}.$$

Given  $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}$ , define

$$(5.2) \quad \text{count}_m(i, \lambda) := \#\{k : \lambda_k > i - m + k - 1\}.$$

The main idea behind (5.2) is that if  $\lambda = \lambda(S)$  is the partition corresponding to a charge  $m$  semi-infinite monomial  $S \in \mathcal{S}_m$ , then

$$(5.3) \quad \text{count}_m(i, \lambda) = \text{count}(i, S),$$

where  $\text{count}(i, S)$  denotes the number of elements of  $S$  which are strictly greater than  $i$ , see (4.6). That (5.3) holds true is easy to check using (4.6) and Proposition 4.1 (b).

**5.2.** We now use (5.2) to define certain combinatorial operators on partitions. Precisely, if  $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}_{m,i}$ , then define  $p_{m,i}(\lambda)$  to be the partition  $\mu = (\mu_1, \mu_2, \dots)$  where:

$$(5.4) \quad \mu_j = \begin{cases} \lambda_j - 1 & \text{for } j \leq \text{count}_m(i, \lambda) \\ i - m + \text{count}_m(i, \lambda) - 1 & \text{for } j = \text{count}_m(i, \lambda) + 1 \\ \lambda_{j-1} & \text{for } j > \text{count}_m(i, \lambda) + 1. \end{cases}$$

On the other hand, if  $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}_{m,i}^*$ , then define  $p_{m,i}^*(\lambda)$  to be the partition  $\mu = (\mu_1, \mu_2, \dots)$  where:

$$(5.5) \quad \mu_j = \begin{cases} \lambda_j + 1 & \text{for } j \leq \text{count}_m(i, \lambda) \\ \lambda_{j+1} & \text{for } j > \text{count}_m(i, \lambda). \end{cases}$$

**5.3.** The following proposition is used in the proof of Proposition 5.2 which relates the combinatorial operators defined in §5.2 to the operators  $f_i f_j^*$  described in §4.7 and (5.1).

**Proposition 5.1.** *Fix  $m, i, j \in \mathbb{Z}$ ,  $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}_{m,j}^*$ , let  $\mu := p_{m,j}^*(\lambda)$ , assume that  $\mu \in \mathcal{P}_{m-1,i}$  and let  $\nu := p_{m-1,i}(\mu) = p_{m-1,i} p_{m,j}^*(\lambda)$ . The following assertions hold true:*

- (a) *if  $i < j$ , then  $\nu \subseteq \lambda$ , the skew diagram  $\lambda \setminus \nu$  is a border strip,  $\#(\lambda \setminus \nu) = j - i$ , and  $\text{height}(\lambda \setminus \nu) = \text{count}_{m-1}(i, \mu) - \text{count}_m(j, \lambda)$ ;*
- (b) *if  $i > j$ , then  $\lambda \subseteq \nu$ , the skew diagram  $\nu \setminus \lambda$  is a border strip,  $\#(\nu \setminus \lambda) = i - j$ , and  $\text{height}(\nu \setminus \lambda) = \text{count}_m(j, \lambda) - \text{count}_{m-1}(i, \mu)$ .*
- (c) *if  $i = j$ , then  $\nu = \lambda$  and the skew diagrams  $\nu \setminus \lambda$  and  $\lambda \setminus \nu$  are empty.*

*Proof.* By assumption we have

$$(5.6) \quad \mu := p_{m,j}^*(\lambda)$$

and

$$(5.7) \quad \nu := p_{m-1,i}(\mu) = p_{m-1,i} p_{m,j}^*(\lambda) = (\nu_1, \nu_2, \dots);$$

set

$$(5.8) \quad \alpha := \text{count}_m(j, \lambda)$$

and

$$(5.9) \quad \beta := \text{count}_{m-1}(i, \mu).$$

For (a), we have  $i < j$ . As a consequence, using the definitions (5.4) and (5.5), we deduce that the partition  $\nu = (\nu_1, \nu_2, \dots)$  has the form:

$$(5.10) \quad \nu_k = \begin{cases} \lambda_k & \text{for } 1 \leq k \leq \alpha \\ \lambda_{k+1} - 1 & \text{for } \alpha + 1 \leq k \leq \beta \\ i - m + \beta & \text{for } k = \beta + 1 \\ \lambda_k & \text{for } k \geq \beta + 2. \end{cases}$$

Considering (5.10), it is clear that  $\nu \subseteq \lambda$ , that  $\theta := \lambda \setminus \nu$  is a border strip, and that the number of rows of  $\theta$  equals

$$(5.11) \quad \#[\alpha + 1, \beta + 1] = \beta - \alpha + 1;$$

it follows from (5.11) that

$$(5.12) \quad \text{height}(\theta) = \beta - \alpha.$$

Next if  $\theta_k$  denotes the number of elements in the  $k$ th row of  $\theta$ , then  $\theta_k = 0$  for  $k \leq \alpha$  and  $k \geq \beta + 2$ . We also have:

$$(5.13) \quad \theta_k = \lambda_k - \lambda_{k+1} + 1,$$

for  $\alpha + 1 \leq k \leq \beta$ ,

$$(5.14) \quad \theta_{\beta+1} = \lambda_{\beta+1} - i - \beta + m,$$

and

$$(5.15) \quad \lambda_{\alpha+1} = j + \alpha - m.$$

Thus, using (5.13), (5.14), and (5.15), we have:

$$\sum_{k=\alpha+1}^{\beta+1} \theta_k = j + \alpha - m - i - \beta + m + \#[\alpha + 1, \beta] = j - i,$$

whence

$$\#\theta = j - i.$$

For (b), we have  $i > j$ . As a consequence, using the definitions (5.4) and (5.5), we deduce that the partition  $\nu = (\nu_1, \nu_2, \dots)$  is defined by:

$$(5.16) \quad \begin{cases} \lambda_k & \text{for } 1 \leq k \leq \beta \\ i + \beta - m & \text{for } k = \beta + 1 \\ \lambda_{k-1} + 1 & \text{for } \beta + 1 < k \leq \alpha + 1 \\ \lambda_k & \text{for } k \geq \alpha + 2. \end{cases}$$

Considering (5.16), it is clear that  $\lambda \subseteq \nu$ , that  $\theta := \nu \setminus \lambda$  is a border strip, and that the number of rows of  $\theta$  equals

$$(5.17) \quad \#[\beta + 1, \alpha + 1] = \alpha - \beta + 1.$$

Thus

$$(5.18) \quad \text{height}(\theta) = \alpha - \beta.$$

Next let  $\theta_k$  denote the number of elements in the  $k$ th row of  $\theta$ . Then  $\theta_k = 0$  for  $k \leq \beta$  and  $k > \alpha + 1$ . We also have:

$$(5.19) \quad \theta_{\beta+1} = i + \beta - m - \lambda_{\beta+1},$$

$$(5.20) \quad \theta_k = \lambda_{k-1} + 1 - \lambda_k,$$

for  $\beta + 1 < k \leq \alpha + 1$ , and

$$(5.21) \quad \lambda_{\alpha+1} = j + \alpha - m.$$

Using (5.19), (5.20), and (5.21), it follows that

$$\sum_{k=\beta+1}^{\alpha+1} \theta_k = i + \beta - m - j - \alpha + m + \#[\beta + 2, \alpha + 1] = i - j$$

so that

$$\#\theta = i - j.$$

Assertion (c) is trivial. □

**5.4. Example.** Recall, see §4.4, that

$$S = (4, 3, 1, 0, -2, -3, -5, -7, -8, \dots)$$

is the element of  $\mathcal{S}_0$  corresponding to the partition

$$\lambda = (4, 4, 3, 3, 2, 2, 1) \in \mathcal{P}_{19},$$

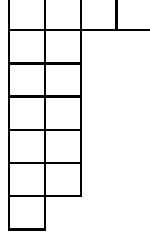
whose Young diagram is pictured in §4.4. To compute  $f_{-1}f_3^*(v_S)$  note that

$$T = (S \setminus \{3\}) \cup \{-1\} = (4, 1, 0, -1, -2, -3, -5, -7, -8, \dots),$$

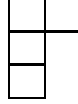
$\text{count}(3, S) = 1$  and  $\text{count}(-1, S \setminus \{3\}) = 3$ . We conclude

$$(5.22) \quad f_{-1}f_3^*(v_S) = (-1)^{3-1}v_T = v_T.$$

To see the combinatorics encoded in (5.22) first note that if  $\nu := \lambda(T)$ , the partition corresponding to  $T$ , then  $\nu = (4, 2, 2, 2, 2, 2, 1)$  which has Young diagram



and  $\nu \subseteq \lambda$ . The skew diagram  $\theta := \lambda \setminus \nu$  is the set  $\{\{2, 3\}, \{2, 4\}, \{3, 3\}, \{4, 3\}\}$  which can be represented pictorially as:



Note that the skew diagram  $\theta$  is a border strip and  $\text{height}(\theta) = 2$ . If we now identify  $S$  with  $\lambda$  and  $T$  with  $\nu$ , then (5.22) takes the form

$$(5.23) \quad f_{-1}f_3^*(v_\lambda) = (-1)^{\text{height}(\theta)}v_\nu.$$

Suppose now that we wish to compute  $f_{-1}f_{-3}^*(v_S)$ . In this case,  $\text{count}(-3, S) = 5$ ,  $\text{count}(-1, S \setminus \{-3\}) = 4$  and hence

$$(5.24) \quad f_{-1}f_{-3}^*(v_S) = -1v_T,$$

where  $T = (4, 3, 1, 0, -1, -2, -5, -7, -8, \dots)$ .

The combinatorics encoded in (5.23) is similar to that encoded in (5.22), but there is one difference which amounts to the fact that  $-1 > -3$  while  $3 > -1$ . In more detail, if  $\nu := \lambda(T)$ , then  $\nu = (4, 4, 3, 3, 3, 3, 1)$ ,  $\lambda \subseteq \nu$ , and the skew diagram  $\theta := \nu \setminus \lambda$  is  $\{\{5, 3\}, \{6, 3\}\}$  which is a border strip. The border strip  $\theta$  can be pictured pictorially as:



and has height equal to one. If we identify  $S$  with  $\lambda$  and  $T$  with  $\nu$ , then (5.24) takes the form

$$f_{-1}f_{-3}^*(v_\lambda) = (-1)^{\text{height}(\theta)}v_\nu.$$

**5.5.** Example 5.4 generalizes:

**Proposition 5.2.** *Suppose that  $S = (s_1, s_2, \dots) \in \mathcal{S}$  and  $i, j \in \mathbb{Z}$ . Then  $f_i f_j^*(v_S) \neq 0$  if and only if  $j \in S$ , and  $i \notin S \setminus \{j\}$ . In addition assume that  $f_i f_j^*(v_S) \neq 0$ , let  $T := (S \setminus \{j\}) \cup \{i\}$ , let  $\lambda$  and  $\nu$  be the partitions determined by  $S$  and  $T$  respectively, and denote  $v_S$  by  $v_\lambda$  and  $v_T$  by  $v_\nu$ . The following assertions hold true:*

(a) *If  $i < j$ , then  $\nu \subseteq \lambda$ , the skew diagram  $\lambda \setminus \nu$  is a border strip of length  $j - i$  and*

$$f_i f_j^*(v_\lambda) = (-1)^{\text{height}(\lambda \setminus \nu)} v_\nu;$$

(b) *If  $j < i$ , then  $\lambda \subseteq \nu$ , the skew diagram  $\nu \setminus \lambda$  is a border strip of length  $i - j$  and*

$$f_i f_j^*(v_\lambda) = (-1)^{\text{height}(\nu \setminus \lambda)} v_\nu.$$

*Proof.* The proposition is a consequence of Proposition 4.1, the discussion given in §5.1 and Proposition 5.1. In particular, using Proposition 4.1 (b) in conjunction with (5.10) and (5.16), depending on whether  $i < j$  or  $j < i$ , we compute that

$$\nu = p_{i,m-1} p_{j,m}^*(\lambda).$$

The conclusion of Proposition 5.2 then follows from Proposition 5.1, (5.2) and (5.1).  $\square$

## 6. THE BOSONIC REPRESENTATION OF $\widehat{\mathfrak{gl}}(\infty)$

We now provide an application of our combinatorial construction given in §5. Indeed, we use this construction to prove the boson-fermion correspondence which we state as Theorem 6.2.

**6.1.** To begin with, let  $A := \mathbb{C}[z, z^{-1}]$  and  $B = A \otimes_{\mathbb{C}} \Lambda := \mathbb{C}[z, z^{-1}, h_1, h_2, \dots]$ . The *bosonic representation of the oscillator algebra* is the Lie algebra homomorphism

$$(6.1) \quad \xi_0 : \mathfrak{s} \rightarrow \text{End}_{\mathbb{C}}(B)$$

determined by:

$$\xi_0((0, t^k)) = p_k^\perp = k \frac{\partial}{\partial p_k}, \text{ for } k > 0;$$

$$\xi_0((0, t^k)) = p_{-k}, \text{ for } k < 0;$$

$$\xi_0((0, 1)) = z \frac{\partial}{\partial z};$$

and

$$\xi_0((1, 0)) = 1,$$

compare with [5, p. 314] or [6, Lecture 5, p. 46].

**6.2.** The first step to proving Theorem 6.2 is to define operators  $b_i \in \text{End}_{\mathbb{C}}(B)$  by the rule:

$$(6.2) \quad b_i(z^m s_\lambda) = \begin{cases} (-1)^{\text{count}_m(i, \lambda)} z^{m+1} s_{p_{m,i}(\lambda)} & \text{for } \lambda \in \mathcal{P}_{m,i} \\ 0 & \text{for } \lambda \notin \mathcal{P}_{m,i}. \end{cases}$$



Similarly define operators  $b_i^* \in \text{End}_{\mathbb{C}}(B)$  by the rule

$$(6.3) \quad b_i^*(z^m s_\lambda) \begin{cases} (-1)^{\text{count}_m(i, \lambda)} z^{m-1} s_{p_{m,i}^*(\lambda)} & \text{for } \lambda \in \mathcal{P}_{m,i}^* \\ 0 & \text{for } \lambda \in \mathcal{P}_{m,i}^* \end{cases}$$

As in (4.9) and (4.10), we have the relations

$$(6.4) \quad b_i b_j^* + b_j^* b_i = \delta_{ij}, \quad b_i b_j + b_j b_i = 0, \quad b_i^* b_j^* + b_j^* b_i^* = 0,$$

and

$$(6.5) \quad [b_i b_j^*, b_\ell b_k^*] = \delta_{j\ell} b_i b_k^* - \delta_{ik} b_\ell b_j^*,$$

for all  $i, j, k, \ell \in \mathbb{Z}$ . Indeed, as in (4.9), (6.4) follows immediately from the definitions while (6.5) is deduced from (6.4).

**6.3. Example.** As in §4.4, if  $\lambda = (4, 4, 3, 3, 2, 2, 1)$ , then  $\text{count}_0(3, \lambda) = 1$  and

$$b_3^*(s_\lambda) = -z^{-1} s_\mu,$$

where  $\mu$  is the partition  $\mu = (5, 3, 3, 2, 2, 1)$ . Also,  $\text{count}_{-1}(-1, \mu) = 3$ ,

$$b_{-1}(z^{-1} s_\mu) = -s_\nu,$$

where  $\nu = (4, 2, 2, 2, 2, 1)$ , and

$$b_{-1} b_3^*(s_\lambda) = s_\nu.$$

**6.4.** The key point in the proof of Theorem 6.2 is the following observation which is a consequence of Proposition 5.2. The point is that if  $S \in \mathcal{S}$ ,  $k$  a nonzero integer and  $\mathfrak{s}_k := (0, t^k) \in \mathfrak{s}$ , then

$$(6.6) \quad \delta_0(\mathfrak{s}_k)(v_S) = \sum_{\text{finite}} f_\ell f_{\ell+k}^*(v_S)$$

and we now give a combinatorial description of this finite set:

**Proposition 6.1.** *Suppose that  $S \in \mathcal{S}_m$ . The following assertions hold true:*

(a) *If  $k > 0$ , then*

$$\mathfrak{s}_k(v_S) = \sum_{\text{finite}} (-1)^{\text{height}(\lambda(S) \setminus \lambda(T))} v_T,$$

*where the finite sum is taken over all  $T \in \mathcal{S}_m$ , which have the property that  $\lambda(T) \subseteq \lambda(S)$ , and  $\lambda(S) \setminus \lambda(T)$  is a border strip of length  $k$ .*

(b) *If  $k < 0$ , then*

$$\mathfrak{s}_k(v_S) = \sum_{\text{finite}} (-1)^{\text{height}(\lambda(T) \setminus \lambda(S))} v_T$$

*where the finite sum is taken over all  $T \in \mathcal{S}_m$  with the property that  $\lambda(S) \subseteq \lambda(T)$  and  $\lambda(T) \setminus \lambda(S)$  is a border strip of length  $|k|$ .*

*Proof.* To begin with, note that for both (a) and (b), Proposition 5.2 implies that each summand of (6.6) contributes a summand of the desired form.

To establish Proposition 6.1 it thus remains to show that, conversely, each border strip of the shape asserted in the proposition appears as a summand of (6.6).

To this end, consider the case that  $k > 0$ . Let  $\lambda$  be the partition corresponding to  $S$ , suppose that  $\nu \subseteq \lambda$  is such that  $\theta := \lambda \setminus \nu$  is a border strip of length  $k$ . Let  $\theta_{n'}$  denote the number of elements in the  $n'$ th row of  $\theta$ . Let  $n := \min\{j : \nu_j \neq \lambda_j\}$ . Then  $\theta_{n'} = 0$  for  $n' < n$  and  $\theta_n \neq 0$ ; set

$$(6.7) \quad \ell := \theta_n - k - n + m + \lambda_{n+1}.$$

We then compute, using the definitions (5.4) and (5.5) together with the fact that  $\theta$  is a border strip, that

$$(6.8) \quad \nu = p_{m-1, \ell} p_{m, \ell+k}^*(\lambda);$$

compare with (5.13) and (5.14).

Thus if  $T$  is the element of  $\mathcal{S}_m$  corresponding to  $\nu$ , then

$$(6.9) \quad (-1)^{\text{height}(\lambda \setminus \nu)} v_T = f_\ell f_{\ell+k}^*(v_S),$$

by Proposition 5.2 (a).

Next suppose that  $k < 0$ . Again let  $\lambda$  be the partition corresponding to  $S$ , suppose that  $\nu \supseteq \lambda$  is such that  $\theta := \nu \setminus \lambda$  is a border strip of length  $|k|$ , let  $T$  be the element of  $\mathcal{S}_m$  corresponding to  $\nu$  and let  $n := \min\{j : \nu_j \neq \lambda_j\}$ . Let  $\theta_{n'}$  denote the number of elements in the  $n'$ th row of  $\theta$  and set

$$(6.10) \quad \ell := \theta_n - (n - 1) + \lambda_n + m.$$

We then compute, using the definitions (5.4) and (5.5) together with the fact that  $\theta$  is a border strip, that:

$$(6.11) \quad \nu = p_{m-1, \ell} p_{m, \ell+k}^*(\lambda);$$

compare with (5.19) and (5.20).

Thus if  $T$  is the element of  $\mathcal{S}_m$  corresponding to  $\nu$ , then

$$(6.12) \quad (-1)^{\text{height}(\nu \setminus \lambda)} v_T = f_\ell f_{\ell+k}^*(v_S),$$

by Proposition 5.2 (b).

□

**6.5.** Using the theory we have developed thus far we can prove the boson-fermion correspondence.

**Theorem 6.2.** *The bosonic representation*

$$\xi_0 : \mathfrak{s} \rightarrow \text{End}_{\mathbb{C}}(B),$$

namely (6.1), of the oscillator algebra extends to a representation

$$\xi : \widehat{\mathfrak{gl}}(\infty) \rightarrow \text{End}_{\mathbb{C}}(B)$$

of the Lie algebra  $\widehat{\mathfrak{gl}}(\infty)$ . More precisely, the Lie algebra  $\widehat{\mathfrak{gl}}(\infty)$  admits a representation  $\xi : \widehat{\mathfrak{gl}}(\infty) \rightarrow \text{End}_{\mathbb{C}}(B)$  with the property that the diagram

$$\begin{array}{ccc} \mathfrak{s} & & \\ \downarrow \delta_0 & \searrow \xi_0 & \\ \widehat{\mathfrak{gl}}(\infty) & \xrightarrow{\xi} & \text{End}_{\mathbb{C}}(B) \end{array}$$

commutes. In addition, the  $\mathbb{C}$ -linear isomorphism

$$\sigma : F \rightarrow B$$

defined by

$$v_S \mapsto z^m s_{\lambda(S)},$$

for  $m = \text{charge}(S)$  and  $\lambda(S)$  the partition determined by the semi-infinite monomial  $S$ , is an isomorphism of  $\widehat{\mathfrak{gl}}(\infty)$ -modules.

*Proof.* Consider the representation  $\xi : \widehat{\mathfrak{gl}}(\infty) \rightarrow \text{End}_{\mathbb{C}}(B)$  determined by the conditions that:

$$\xi((0, E_{ij})) = \begin{cases} b_i b_j^* & \text{if } i \neq j \text{ or } i = j > 0 \\ b_i b_i^* - \text{id}_B & \text{if } j = i \leq 0 \end{cases}$$

and

$$\xi((a, 0)) = a \text{id}_B,$$

for  $a \in \mathbb{C}$ . The fact that  $\xi$  is a representation follows from the relations given in (6.4). The fact that  $\xi$  extends the representation  $\xi_0$  follows from Proposition 6.1 and the Murnaghan-Nakayama rule (2.1) and (2.2).

For the second assertion, fix  $i, j \in \mathbb{Z}$  and assume that  $f_i f_j^*(v_S) \neq 0$ . We then have that

$$f_i f_j^*(v_S) = (-1)^\alpha v_T,$$

where

$$T := (S \setminus \{j\}) \cup \{i\},$$

and

$$\alpha := \text{count}(i, S \setminus \{j\}) - \text{count}(j, S);$$

let  $\lambda = \lambda(S)$  be the partition corresponding to  $S$  and let  $\nu = \lambda(T)$  be the partition corresponding to  $T$ .

In this setting, the operator  $p_{m,j}^*$  is defined on the partition  $\lambda$  and the operator  $p_{m-1,i}$  is defined on the partition  $p_{m,j}^*(\lambda)$ . In addition

$$\nu = p_{m-1,i} p_{m,j}^*(\lambda).$$

On the other hand we have that

$$\sigma(v_S) = z^m s_\lambda.$$

Considering the definitions of the operators  $b_i$  and  $b_j^*$ , we then deduce that

$$b_i b_j^*(z^m s_\lambda) = (-1)^\alpha z^m s_\nu$$

which is what we wanted to show. □

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